

*9th Annual Eastern Oregon University  
Mathematics Competition Exam – 2012*

1. We can make use of the identity  $1 + 2 + \dots + n = \frac{n(n+1)}{2}$ . First note that the number at coordinates  $(1, j)$  is whatever comes after listing the numbers in the diagonals below that point. Those diagonals have  $1, 2, 3, \dots, (j - 1)$  numbers in them, respectively, to they include all the numbers up to and including  $\frac{(j-1)(j)}{2}$ . Therefore the number at coordinates  $(1, j)$  will be the next integer  $\frac{(j-1)(j)}{2} + 1$ .

The number at coordinates  $(i, j)$  is the  $i^{\text{th}}$  number in the diagonal that began at  $(1, j + i - 1)$ . So the value of that number will be  $\frac{((j+i-1)-1)(j+i-1)}{2} + i$ . Multiplied out and simplified, this means that the value at coordinates  $(i, j)$  will be  $\frac{j^2 + 2ij + i^2 - 3j - i + 2}{2}$ .

2. Let  $R \geq 3$  be a fixed positive integer and let  $0 \leq x \leq 1$ .

Let  $f(x) = (R + x)^{R-x} = e^{(R-x)\ln(R+x)}$ .

So,  $f'(x) = (R + x)^{R-x} \cdot \left(\frac{R-x}{R+x} - \ln(R+x)\right) \leq (R + x)^{R-x} \cdot (1 - \ln(R+x)) < 0$  with the latter inequality due to the fact that  $R + x > e$ .

This means  $f(x)$  is a decreasing function when  $0 < x < 1$ , and so  $f(1) < f(0)$ .

Therefore  $(R + 1)^{R-1} < R^R$  whenever  $R \geq 3$ .

In particular  $101^{99}$  is less than  $100^{100}$ .

3. The 720 rearrangements can be grouped to see that each of the six digits appears in each decimal place 120 times. Therefore the sum of each decimal place will be:

$$120 * (1 + 2 + 3 + 4 + 5 + 6) = 120 * 21 = 2,520.$$

That sum occurs in each of the six decimal places, so the total sum of all 720 rearrangements is:

$$2,520 * (1 + 10 + 100 + 1000 + 10000 + 100000) = 2,520 * 111,111 = 279,999,720.$$

4. Subtract  $\frac{2}{x-1}$  from both sides of the equation and put the fractions over a common denominator to find  $\frac{3x-7}{(x+1)(x-1)} < 0$ . For three quantities to have a negative product, either one or three of them must be negative. This gives solutions in two ranges:  $x < -1$  or  $1 < x < \frac{7}{3}$ .

5. Suppose there were positive solutions. Cube the equation involving squares and square the equation involving cubes. This gives:

$$a^6 + 3a^4b^2 + 3a^2b^4 + b^6 = c^6 = a^6 + 2a^3b^3 + b^6.$$

Now subtract  $a^6 + b^6$  from both side and divide what is left by  $a^2b^2$  (which is allowable since we're assuming they are all positive numbers). This gives

$$3a^2 + 3b^2 = 2ab \text{ which we can rewrite as } 3a^2 - 2ab + 3b^2 = 0.$$

We can solve this as a quadratic in  $a$  and find  $a = \frac{2b \pm \sqrt{4b^2 - 36b^2}}{6}$ . Since this gives an imaginary result anytime  $b$  is positive, we can't have a solution where  $a$  and  $b$  are both positive real numbers.

6. A polynomial with real coefficients can't have only one complex root, so for  $p(x)$  to have exactly two real roots, it must be of the form  $p(x) = (x - a)(x - b)^2$  where  $a$  and  $b$  are the roots. Therefore  $p(x) = 2(x - a)(x - b) + (x - b)^2$ . In particular,  $p'(x)$  is divisible by  $(x - b)$ , and so  $b$  must be a root of  $p'(x)$  as well as being a root of  $p(x)$ . But  $p'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 6(x - 3)(x + 1)$ . So the only roots of  $p'(x)$  are  $x = -1$  and  $x = 3$ . For  $x = -1$  to be a root of  $p(x)$  it must be that  $0 = p(-1) = 5 + C$ . So  $C = -5$ . Similarly for  $x = 3$  to be a root of  $p(x)$ , we need  $0 = p(3) = -27 + C$ . So  $C = 27$ . It is now routine to check that  $p(x) = x^3 - 3x^2 - 9x - 5 = (x + 1)^2(x - 5)$ , which has exactly two roots. Likewise,  $p(x) = x^3 - 3x^2 - 9x + 27 = (x - 3)^2(x + 3)$ , which also has exactly two roots. So  $C = -5$  and  $C = 27$  are exactly the constants that give  $p(x)$  two real roots.

7. Let  $f(x) = x - \sin(x)$ . First note that  $f(0) = 0$ . Now  $f'(x) = 1 - \cos(x) \geq 0$  for all values of  $x$ . Therefore  $f(x)$  is non-decreasing, so  $f(x) \geq 0$  whenever  $x \geq 0$ . As a consequence  $x \geq \sin(x)$  for all non-negative  $x$ .

Restricting to  $0 \leq r \leq 1$  we know:

$$\sin^2(r) + \cos^2(r) = 1.$$

$$r^2 + \cos^2(r) \geq 1.$$

$$\cos^2(r) \geq 1 - r^2.$$

$$\cos(r) \geq \sqrt{1 - r^2}.$$

Since  $\cos(0) = 1$  and since  $\sqrt{1 - 0^2} = 1$  we can integrate to get the inequality:

$$\int_0^x \cos(r) \, dr \geq \int_0^x \sqrt{1 - r^2} \, dr$$

$$\sin(x) \geq \int_0^x \sqrt{1 - r^2} \, dr$$