THE FIRST L^p-COHOMOLOGY OF SOME FINITELY GENERATED GROUPS AND p-HARMONIC FUNCTIONS

MICHAEL J. PULS

Abstract. Let G be a finitely generated infinite group and let $p > 1$. In this paper we make a connection between the first L^p-cohomology space of G and p-harmonic functions on G. We also describe the elements in the first L^p-cohomology space of groups with polynomial growth, and we give an inclusion result for nonamenable groups.

1. Introduction

In this paper G will always be a finitely generated infinite group and S will always denote a symmetric generating set for G. Let M be a right G-module and let $C^n(G,M)$ be the set of functions from $G^n = G \times \cdots \times G(n$ copies) to M. We now have a chain complex

$$0 \longrightarrow M \xrightarrow{\delta_0} C^1(G,M) \xrightarrow{\delta_1} C^2(G,M) \xrightarrow{\delta_2} \cdots \xrightarrow{\delta_{n-1}} C^n(G,M) \xrightarrow{\delta_n} \cdots$$

where for $n \geq 0$

$$(\delta_n f)(g_1,\dots,g_{n+1}) = (f(g_2,\dots,g_{n+1}))\cdot g_1 +$$
$$\sum_{k=1}^{n}(-1)^k f(g_1,\dots,g_kg_{k+1},\dots,g_{n+1}) + (-1)^{n+1}f(g_1,\dots,g_n),$$

where g_kg_{k+1} occupies the kth position of f. Each δ_n is a linear map and a calculation shows $\delta_n\delta_{n-1} = 0$. The nth group cohomology of G with coefficients in M is denoted by $H^n(G,M)$ and is equal to ker δ_n/im δ_{n-1}, where ker δ_n denotes the kernel of δ_n and im δ_{n-1} is the image of δ_{n-1} in $C^n(G,M)$. If M is a Banach space then we can give $C^n(G,M)$ the compact open topology. This means $f_k \to f$ in $C^n(G,M)$ if and only if $f_k(g_1,\dots,g_n) \to f(g_1,\dots,g_n)$. Note that ker δ_n is a Banach space. In general im δ_{n-1} is not closed in ker δ_n. Let $\overline{\text{im }\delta_{n-1}}$ denote the closure of im δ_{n-1} in ker δ_n. The quotient space $\overline{H}^n(G) = \ker \delta_n/\overline{\text{im }\delta_{n-1}}$ is called the nth reduced cohomology space of M.

Let $\mathcal{F}(G)$ denote the set of real-valued functions on G. Let $f \in \mathcal{F}(G)$ and let $x \in G$, the right translation of f by x is the function defined by $f_x(g) = f(gx^{-1})$. For a real number $p \geq 1, L^p(G)$ will consist of those $f \in \mathcal{F}(G)$ for which $\sum_{g\in G}|f(g)|^p < \infty$, and $C_0(G)$ will consist of those $f \in \mathcal{F}(G)$ for which the set $\{g \mid |f(g)| > \epsilon\}$ is finite for each $\epsilon > 0$. The sets $\mathcal{F}(G), C_0(G)$ and $L^p(G)$ are G-modules under right translations. In this paper we study $H^1(G,L^p(G))$, the first

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group cohomology of G with coefficients in $L^p(G)$. We also study $\overline{H}^1_{(p)}(G)$, the first reduced L^p-cohomology space.

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2. Preliminaries

Let $\mathbb{R}G$ be the group ring of G over $\mathbb{R}$. For $h \in \mathbb{R}G$ and $f \in \mathcal{F}(G)$ we define a multiplication from $\mathcal{F}(G) \times \mathbb{R}G$ into $\mathcal{F}(G)$ by

$$(f * h)(g) = \sum_{x \in G} f(gx^{-1})h(x).$$

Observe that $(f * (s-1))(g) = f(gs^{-1}) - f(g)$ for each $g \in G$ and each $s \in S$. For $1 \leq p \in \mathbb{R}$, let $D^p(G) = \{f \in \mathcal{F}(G) \mid f * (s-1) \in L^p(G) \text{ for each } s \in S\}$. Recall that S is a symmetric set of generators for G. We define a norm on $L^p(G)$ by $\| f \|_p = (\sum_{g \in G} |f(g)|^p)^{\frac{1}{p}}$, where $f \in L^p(G)$. Let $h \in D^p(G)$ and let e be the identity element of G. We define a norm on $D^p(G)$ by $\| h \|_{D^p(G)} = \left(\sum_{s \in S} \| h * (s-1) \|_p^p + |h(e)|^p\right)^{\frac{1}{p}}$. Under this norm $D^p(G)$ is a Banach space. Let f_1 and f_2 be elements of $D^p(G)$. We will write $f_1 \simeq f_2$ if $f_1 - f_2$ is a constant function. Clearly $\simeq$ is an equivalence relation on $D^p(G)$. Identify the constant functions on G with $\mathbb{R}$. Now $D^p(G)/\mathbb{R}$ is a Banach space under the norm induced from $D^p(G)$. That is, if $[f]$ is an equivalence class from $D^p(G)/\mathbb{R}$ then $\| [f] \|_{D^p(G)/\mathbb{R}} = \left(\sum_{s \in S} \| f * (s-1) \|_p^p\right)^{\frac{1}{p}}$. We shall write $\| f \|_{D(p)}$ for $\| [f] \|_{D^p(G)/\mathbb{R}}$. The norm for $D^p(G)$ and $D^p(G)/\mathbb{R}$ depends on the symmetric generating set S, but the underlying topology does not. If $X \subset D^p(G)$, then $(\overline{X})_{D^p(G)}$ will denote the closure of X in $D^p(G)$. Similarly if $Y \subset D^p(G)/\mathbb{R}$, then $(\overline{Y})_{D(p)}$ will denote the closure of Y in $D^p(G)/\mathbb{R}$. The cardinality of a set A will be denoted by $|A|$.

For the chain complex

$$0 \longrightarrow L^p(G) \xrightarrow{\delta_0} C^1(G, L^p(G)) \xrightarrow{\delta_1} C^2(G, L^p(G)) \xrightarrow{\delta_2} \cdots$$

the map δ_0 is given by $(\delta_0 f)(g) = f * (g-1)$. If $f \in \mathcal{F}(G)$ and $\delta_0 f \in C^1(G, L^p(G))$, then $\delta_1(\delta_0 f) = 0$ which implies $D^p(G)/\mathbb{R} \subseteq \ker \delta_1$ since $f * (g-1) \in L^p(G)$ for all $g \in G$. We now show that the reverse inclusion is also true.

Lemma 2.1. *Let G be a finitely generated, infinite group. The kernel of $\delta_1 : C^1(G, L^p(G)) \to C^2(G, L^p(G))$ is $D^p(G)/\mathbb{R}$.*

Proof. Consider the chain complex

$$0 \longrightarrow \mathcal{F}(G) \xrightarrow{\delta_0'} C^1(G, \mathcal{F}(G)) \xrightarrow{\delta_1'} C^2(G, \mathcal{F}(G)) \xrightarrow{\delta_2'} \cdots$$

It was shown in [9, Lemma 4.2] that $H^1(G, \mathcal{F}(G)) = 0$. Thus for each $f \in \ker \delta_1'$ there exists $\bar{f}$ in $\mathcal{F}(G)$ such that $\delta_0' \bar{f} = f$.

Let $f_1 \in C^1(G, L^p(G)) \subseteq C^1(G, \mathcal{F}(G))$ and suppose $\delta_1 f_1 = 0$. Then $\delta_1' f_1 = 0$ which implies there exists an $\bar{f_1} \in \mathcal{F}(G)$ such that $\delta_0 \bar{f_1} = f_1$. The result now follows since $\bar{f_1} \in D^p(G)$. □

The map δ_0 is an injection so we obtain the following:

(a) The first cohomology group of G with coefficients in $L^p(G)$, denoted by $H^1(G, L^p(G))$, is isomorphic with $D^p(G) / (L^p(G) \bigoplus \mathbb{R})$.

(b) The first reduced L^p-cohomology space of G, denoted by $\overline{H}^1_{(p)}(G)$, is isometric with $D^p(G)/\overline{L^p(G)\bigoplus\mathbb{R}}$, where the closure is taken in $D^p(G)$.

3. Nonreduced L^p-cohomology and p-harmonic functions

Let $f \in \mathcal{F}(G)$ and let $g \in G$. Suppose $1 < p \in \mathbb{R}$ and define

$$\triangle_p f(g) := \sum_{s\in S} |f(gs^{-1}) - f(g)|^{p-2}(f(gs^{-1}) - f(g)).$$

In the case $1 < p < 2$, we make the convention that $|f(gs^{-1}) - f(g)|^{p-2}(f(gs^{-1}) - f(g)) = 0$ if $f(gs^{-1}) = f(g)$. We shall say that f is *p-harmonic* if $f \in D^p(G)$ and $\triangle_p f(g) = 0$ for all $g \in G$. Recall that f is a harmonic function if $\sum_{s\in S}(f(gs^{-1}) - f(g)) = 0$ for all $g \in G$. Let $HD^p(G)$ be the set of p-harmonic functions on G. Observe that the constant functions are in $HD^p(G)$. If $p = 2$, then $HD^2(G)$ is the linear space of harmonic functions on G with finite energy. In general, $HD^p(G)$ is not a linear space if $p \neq 2$. A wealth of information about p-harmonic functions on graphs and manifolds can be found in [6, 7]. Many of the ideas in this section come from the paper [16]. In this section we will give a decomposition of $D^p(G)$ that will allow us to make a connection between p-harmonic functions on G and $\overline{H}^1_{(p)}(G)$.

We begin by giving some preliminary results for both $D^p(G)$ and $HD^p(G)$. Let f and h be elements in $D^p(G)$ and let $1 < p \in \mathbb{R}$. Define

$$\begin{aligned}\langle\triangle_p h, f\rangle &:= \sum_{g\in G}\sum_{s\in S} |h(gs^{-1}) - h(g)|^{p-2}(h(gs^{-1}) - h(g))(f(gs^{-1}) - f(g))\\ &= \sum_{g\in G}\sum_{s\in S} |(h*(s-1))(g)|^{p-2}((h*(s-1))(g))((f*(s-1))(g)).\end{aligned}$$

The above sum exist since $\sum_{g\in G}\sum_{s\in S}\left||h(gs^{-1}) - h(g)|^{p-2}(h(gs^{-1}) - h(g))\right|^q < \infty$, where $\frac{1}{p} + \frac{1}{q} = 1$. The next lemma will be used to help show the uniqueness of the decomposition of $D^p(G)$ that will be given in Theorem 3.5.

Lemma 3.1. *Let f_1 and f_2 be functions in $D^p(G)$. Then $\langle\triangle_p f_1 - \triangle_p f_2, f_1 - f_2\rangle = 0$ if and only if $f_1 * (s-1) = f_2 * (s-1)$ for all $s \in S$.*

Proof. Let $f_1, f_2 \in D^p(G)$ and assume there exists $s \in S$ such that $f_1 * (s-1) \neq f_2 * (s-1)$. Define a function $f : [0,1] \to \mathbb{R}$ by $f(t) = \sum_{g\in G}\sum_{s\in S} |f_1(gs^{-1}) - f_1(g) + t((f_2(gs^{-1}) - f_2(g)) - (f_1(gs^{-1}) - f_1(g)))|^p$. Observe that $f(0) = \| f_1 \|_{D(p)}$ and $f(1) = \| f_2 \|_{D(p)}$. A derivative calculation gives $\frac{df}{dt}|_{t=0} = p\langle\triangle_p f_1, f_2 - f_1\rangle = -p\langle\triangle_p f_1, f_1 - f_2\rangle$. It follows from Proposition 5.4 on page 24 of [3] that $\| f_2 \|_{D(p)} > \| f_1 \|_{D(p)} - p\langle\triangle_p f_1, f_1 - f_2\rangle$. Similarly, $\| f_1 \|_{D(p)} > \| f_2 \|_{D(p)} - p\langle\triangle_p f_2, f_2 - f_1\rangle$. Hence, $p\langle\triangle_p f_1 - \triangle_p f_2, f_1 - f_2\rangle > 0$ if there exists $s \in S$ such that $f_1 * (s-1) \neq f_2 * (s-1)$.

Conversely, suppose $f_1 * (s-1) = f_2 * (s-1)$ for all $s \in S$. Then $\langle\triangle_p f_1 - \triangle_p f_2, f_1 - f_2\rangle = 0$ since $f_1 - f_2$ is a constant function on G. $\square$

For $g \in G$, define δ_g by $\delta_g(x) = 0$ if $x \neq g$ and $\delta_g(g) = 1$.

Lemma 3.2. *Let $h \in \mathcal{F}(G)$. Then h is a p-harmonic function if and only if $\langle\triangle_p h, \delta_x\rangle = 0$ for all $x \in G$.*

Proof. Let $x \in G$ and suppose $h \in HD^p(G)$, then

$$\langle \triangle_p h, \delta_x \rangle = -2 \sum_{s \in S} |h(xs^{-1}) - h(x)|^{p-2}(h(xs^{-1}) - h(x)) = 0.$$

Conversely, if $\langle \triangle_p h, \delta_x \rangle = 0$ for all $x \in G$, then $\sum_{s\in S} |h(xs^{-1}) - h(x)|^{p-2}(h(xs^{-1}) - h(x)) = 0$ for all $x \in G$. □

Remark 3.3. It follows immediately from the lemma that if $h \in HD^p(G)$, then $\langle \triangle_p h, f\rangle = 0$ for all $f \in \mathbb{R}G$.

Proposition 3.4. *If $h \in HD^p(G)$ and $f \in (\overline{L^p(G)})_{D^p(G)}$, then $\langle \triangle_p h, f \rangle = 0$.*

Proof. Let $f \in (\overline{L^p(G)})_{D^p(G)}$ and let $h \in HD^p(G)$. There exists a sequence $\{f_n\}$ in $\mathbb{R}G$ such that $\| f - f_n \|_{D^p(G)} \to 0$ as $n \to \infty$, since $(\overline{\mathbb{R}G})_{D^p(G)} = (\overline{L^p(G)})_{D^p(G)}$. Now

$$\begin{aligned}
0 &\le |\langle \triangle_p h, f\rangle| \\
&= |\langle \triangle_p h, f - f_n\rangle| \\
&= |\sum_{g\in G}\sum_{s\in S} |(h*(s-1))(g)|^{p-2}((h*(s-1))(g))(((f-f_n)*(s-1))(g))| \\
&\le \sum_{g\in G}\sum_{s\in S} |(h*(s-1))(g)|^{p-1}|((f-f_n)*(s-1))(g)| \\
&\le (\sum_{g\in G}\sum_{s\in S}(|(h*(s-1))(g)|^{p-1})^q) \| f_n - f \|_{D(p)} \to 0
\end{aligned}$$

as $n \to \infty$. The last inequality follows from Hölder's inequality □

We now give a decomposition of $D^p(G)$ that will allow us to determine representatives for the nonzero classes in $\overline{H}^1_{(p)}(G)$.

Theorem 3.5. *Let $1 < p \in \mathbb{R}$ and suppose $\overline{L^p(G)}_{D^p(G)} \neq D^p(G)$. If $f \in D^p(G)$, then we can write $f = u + h$, where $u \in (\overline{\mathbb{R}(G)})_{D^p(G)}$ and $h \in HD^p(G)$. This decomposition is unique up to a constant function.*

Proof. Let $f \in D^p(G)$ and let r equal the distance of f from $(\overline{\mathbb{R}G})_{D^p(G)}$ in the $D^p(G)$-norm. Set $B = \{v \in (\overline{\mathbb{R}G})_{D^p(G)} \mid \| f - v \|_{D^p(G)} \le r + 1\}$. Now B is a nonempty weakly compact set in the reflexive Banach space $D^p(G)$. The function $F(v) = \| f - v \|_{D^p(G)}$ is a weakly lower semi-continuous function on B, so $F(v)$ assumes a minimum value on B. This minimum must be r. Let $u \in (\overline{\mathbb{R}G})_{D^p(G)}$ where $\| f - u \|_{D^p(G)} = r$ and set $h = f - u$. We now proceed to show that h is p-harmonic on G. Let $t \in \mathbb{R}$ and let $w \in \mathbb{R}G$. Now, $\| h \|_{D^p(G)} = \| f - u \|_{D^p(G)} \le \| f - (u - tw) \|_{D^p(G)}$ for all $t \in \mathbb{R}$. The minimum of $\| f - (u - tw) \|_{D^p(G)}$ occurs when $t = 0$. Thus $\frac{d}{dt}(\| h + tw \|_{D^p(G)}) \mid_{t=0} = \sum_{g\in G}\sum_{s\in S} p|h(gs^{-1}) - h(g)|^{p-2}(h(gs^{-1}) - h(g))(w(gs^{-1}) - w(g)) = 0$. Using δ_x for w in the above derivative calculation we obtain $-2p\sum_{s\in S} |h(xs^{-1}) - h(x)|^{p-2}(h(xs^{-1}) - h(x)) = 0$. Thus h is p-harmonic by Lemma 3.2.

We now show that this decomposition is unique up to a constant. Suppose $f = u_1 + h_1 = u_2 + h_2$, where $u_1, u_2 \in (\overline{\mathbb{R}G})_{D^p(G)}$ and $h_1, h_2 \in HD^p(G)$. Now, $\langle \triangle_p h_1 - \triangle_p h_2, h_1 - h_2\rangle = \langle \triangle_p h_1 - \triangle_p h_2, u_2 - u_1\rangle = 0$ by Proposition 3.4 since $u_2 - u_1 \in (\overline{\mathbb{R}G})_{D^p(G)}$. By Lemma 3.1, $h_1 * (s - 1) = h_2 * (s - 1)$ for all $s \in S$.

Thus $h_1 - h_2$ is a constant function, which implies that $u_1 - u_2$ is also a constant function. □

We saw in section 2 that $\overline{H}^1_{(p)}(G) = D^P(G)/\overline{L^p(G) \bigoplus \mathbb{R}}$, so it follows from the theorem that each nonzero class in $\overline{H}^1_{(p)}(G)$ can be represented by a nonconstant element in $HD^p(G)$. Thomas Schick has found an error in the proof of Theorem 5.3 from [10]. We were unable to fix the proof of that theorem using the techniques of [10]. However the statement of the theorem is correct since it is a corollary of

Corollary 3.6. *If G is a finitely generated group with polynomial growth, then $\overline{H}^1_{(p)}(G) = 0$ for $1 < p \in \mathbb{R}$.*

Proof. It was shown in [8, Corollary 1.10] that $HD^p(G) = \mathbb{R}$ if G has polynomial growth. The result now follows. □

4. Nonamenable groups and L^p-cohomology

In this section we will give some results concerning the first L^p-cohomology space of nonamenable groups.

Let A be a subset of a group G and define $\partial A := \{x \in A \mid$ there exists $s \in S$ with $xs \notin A\}$. We shall say that a group G is *amenable* if it has an exhaustion $G_1 \subset G_2 \subset \cdots, \cup_{k=1}^{\infty} G_k = G$ by finite subsets such that $\lim_{k\to\infty} \frac{|\partial G_k|}{|G_k|} = 0$. A group that is not amenable is said to be *nonamenable*. Our first result shows how amenability affects the way $L^p(G)$ sits inside $D^p(G)/\mathbb{R}$.

Theorem 4.1. *Let G be a finitely generated infinite group and let $1 \leq p \in \mathbb{R}$. Then $L^p(G)$ is closed in $D^p(G)/\mathbb{R}$ if and only if G is nonamenable.*

Proof. Assume that G is nonamenable and let $f \in (\overline{L^p(G)})_{D(p)} = (\overline{\mathbb{R}G})_{D(p)}$. Now there exists a sequence $\{f_n\}$ in $\mathbb{R}G$ such that $f_n \to f$ in the Banach space $D^p(G)/\mathbb{R}$. Thus $\{f_n\}$ is a Cauchy sequence in $D^p(G)/\mathbb{R}$. It was shown in [4] that there exist a constant C_p such that $\parallel u \parallel_p \leq C_p \parallel u \parallel_{D(p)}$ for all $u \in \mathbb{R}G$ if and only if G is a finitely generated nonamenable group. Thus $\{f_n\}$ is also a Cauchy sequence in $L^p(G)$. So there exists a $\bar{f} \in L^p(G)$ such that $\parallel \bar{f} - f_n \parallel_p \to 0$ as $n \to \infty$. Since L^p-convergence implies pointwise convergence we have that $\parallel (\bar{f} - f_n) * (s-1) \parallel_p \to 0$ as $n \to \infty$ for each $s \in S$. Hence $\parallel \bar{f} - f_n \parallel_{D(p)} \to 0$ as $n \to \infty$. Therefore, $\bar{f} = f$.

Conversely, suppose there exists an exhaustion $G_1 \subseteq G_2 \subseteq \cdots, \cup_{k=1}^{\infty} G_k = G$ by finite subsets such that $\lim_{k\to\infty} \frac{|\partial G_k|}{|G_k|} = 0$. Let χ_k denote the characteristic function on G_k. Now define a function f_k on G by $f_k := \frac{\chi_k}{\sqrt[p]{|G_k|}}$. Note that $\parallel f_k \parallel_p = 1$ for all k. Computing the $D^p(G)/\mathbb{R}$-norm of f_k we obtain $\parallel f_k \parallel^p_{D(p)} = \sum_{s\in S} \parallel f_k * (s-1) \parallel^p_p \leq 2\frac{|\partial G_k|}{|G_k|}$. Thus, $\lim_{k\to\infty} \parallel f_k \parallel_{D(p)} \to 0$ and $\parallel f_k \parallel_p = 1$ for all k. Therefore, if G is amenable, then $L^p(G)$ is not closed in $D^p(G)/\mathbb{R}$. □

The next result follows immediately from the theorem.

Corollary 4.2. *Let $1 \leq p \in \mathbb{R}$. If G is a finitely generated nonamenable group, then $H^1(G, L^p(G)) = \overline{H}^1_{(p)}(G)$ and $H^1(G, L^p(G))$ is a Banach space.*

We will now use the theorem to show $\overline{H}^1_{(p)}(G) \neq 0$ for groups with infinitely many ends.

Corollary 4.3. *Let $1 < p \in \mathbb{R}$. If G is a finitely generated group with infinitely many ends, then $\overline{H}^1_{(p)}(G) \neq 0$.*

Proof. We will prove the corollary by constructing a nonconstant harmonic function, say h, on G that is an element of $D^p(G)$. It will then follow from the maximum principle for harmonic functions that $h \notin C_0(G)$. By the theorem $h \notin (\overline{L^p(G)})_{D^p(G)}$ since $L^p(G) \subset C_0(G)$. Theorem 3.5 then shows $h = u + v$, where $u \in \overline{L^p(G)}_{D^p(G)}$ and v is a nonconstant p-harmonic function, thus h will represent a nonzero class in $\overline{H}^1_{(p)}(G)$. We now proceed to construct h by using a technique that was used in the proof of Theorem 4.1 of [11].

Recall that S is a symmetric generating set for G. Define an element P in $\mathbb{R}G$ by $P = \frac{1}{|S|}\sum_{s \in S} s$. Let $f \in L^p(G)$ and define a bounded linear operator on $L^p(G)$ by $P(f) = f * P$. Observe that f is a harmonic function on G if $f * (P - I) = 0$, where I is the identity operator on $L^p(G)$. Let $k \in \mathbb{N}$ and denote by P^k multiplication of P with itself k times. Since G is nonamenable, $\parallel P \parallel < 1$ in the operator norm for $1 < p \in \mathbb{R}$ [4]. Thus $F = -\sum_{k=0}^{\infty} P^k$ is a bounded operator on $L^p(G)$. Note F is the inverse of $P - I$ in the space of bounded linear operators on $L^p(G)$. Let X denote the Cayley graph of G with respect to a the generating set S. Thus the vertices of X are the elements of G, and $g_1, g_2 \in G$ are joined by an edge if and only if $g_1 = g_2 s^{\pm 1}$ for some generator s. Remove a finite number of vertices and edges of X to obtain two disjoint, infinite, connected components of X. Let X_1 and X_2 denote the components. Define a function h_1 on G by

$$h_1(g) = \begin{cases} 2 & \text{if g is a vertex in } X_1 \\ 1 & \text{if g is a vertex in } X_2 \\ 0 & \text{otherwise} \end{cases} .$$

The support of $h_1 * (P - I)$ is contained in $\partial X_1 \bigcup \partial X_2$, so $h_1 * (P - I) \in \mathbb{R}G$. Thus $F(h_1 * (I - P)) \in L^p(G)$. Set $h_2 = F(h_1 * (I - P))$ and let $h = h_1 - h_2$. Note $h \in D^p(G)$ since $h_1 \in D^p(G)$ and $h_2 \in L^p(G)$. Now, $h * (P - I) = h_1 * (P - I) - h_1 * (P - I) = 0$, so h is a harmonic on G. □

Alain Valette has pointed out a simpler, but different, proof of the above corollary. We now proceed to give a quick sketch of his proof. By combining Lemma 2 with the remark after Proposition 1 of [1] we have that the dimension of $H^1(G, \mathbb{R}G)$ over $\mathbb{R}$ is the number of ends of G minus one. The corollary now follows from Theorem 4.1. Valette's proof also shows that the corollary is true for $p = 1$.

The situation becomes unclear if G is nonamenable with one end. For example, $H^1(SL_n(\mathbb{Z}), L^2(SL_n(\mathbb{Z}))) = 0$ for $n \geq 3$ since $SL_n(\mathbb{Z})$ has property T when $n \geq 3$ [5]. On the other hand, if G is a fundamental group of a closed Riemann surface of genus at least 2, then $H^1(G, L^2(G)) \neq 0$ [2]. A good deal of information about $H^1(G, L^2(G))$ can be found in [1].

5. A Description of $H^1(G, L^p(G))$

In this section we will describe the nonzero elements of $H^1(G, L^p(G))$ for groups that have polynomial growth of (precise) degree $n > p$. These results are a generalization of results from Section 6 of [10], where $H^1(G, L^2(G))$ was discussed.

Let $d > 1$. We shall say that G satisfies condition S_d if there exists a constant $C > 0$ such that $\parallel f \parallel_{\frac{d}{d-1}} \leq C \parallel f \parallel_{D(1)}$ for all $f \in \mathbb{R}G$. If $f \in \mathcal{F}(G)$ and $t \geq 1$, then f^t will denote the function $f^t(g) = (f(g))^t$. The following was proved in [10] but we include it here for completeness.

Lemma 5.1. *Let G be a finitely generated group and let t be a real number greater than or equal to 2. If f is a non-negative real-valued function in $\mathcal{F}(G)$, then*

$$\| f^t \|_{D(1)} \leq 2t \sum_{g\in G} f^{t-1}(g) \left(\sum_{s\in S} |(f * (s-1))(g)| \right).$$

Proof. Let $g \in G$ and let $s \in S$. It follows from the Mean Value Theorem applied to x^t that $(r^t - s^t) \leq t(r^{t-1} + s^{t-1})(r - s)$ where r and s are real numbers with $0 \leq s \leq r$. Thus $|f^t(gs^{-1}) - f^t(g)| \leq t(f^{t-1}(g) + f^{t-1}(gs^{-1}))|f(gs^{-1}) - f(g)|$. We now obtain $\| f^t \|_{D(1)} = \sum_{g\in G}\sum_{s\in S} |(f^t * (s-1))(g)| \leq t \sum_{g\in G}\sum_{s\in S}(f^{t-1}(g) + f^{t-1}(gs^{-1}))|f(gs^{-1}) - f(g)| = 2t \sum_{g\in G}\sum_{s\in S} f^{t-1}(g)|f(gs^{-1}) - f(g)|$ □

The next proposition is a generalization of Proposition 6.2 of [10]. I would like to thank Thomas Schick for showing me this generalization.

Proposition 5.2. *Let $2 \geq p \in \mathbb{R}$ and let $d > p$. If G satisfies condition S_d, then there is a constant $C' > 0$ such that $\| f \|_{\frac{pd}{d-p}} \leq C' \| f \|_{D(p)}$ for all $f \in \mathbb{R}G$.*

Proof. Set $t = \frac{pd-p}{d-p}$. By property S_d, Lemma 5.1 and Hölder's inequality we have (assuming without loss of generality that f is non-negative).

$$\begin{aligned}
\| f^{\frac{pd-p}{d-p}} \|_{\frac{d}{d-1}} &\leq C \| f^{\frac{pd-p}{d-p}} \|_{D(1)} \\
&\leq 2C \left(\frac{pd-p}{d-p}\right) \sum_{g\in G} f^{\frac{d(p-1)}{d-p}}(g) \left(\sum_{s\in S} |\big(f * (s-1)\big)(g)| \right) \\
&= 2C \left(\frac{pd-p}{d-p}\right) \sum_{g\in G}\sum_{s\in S} f^{\frac{d(p-1)}{d-p}}(g)|f(gs^{-1}) - f(g)| \\
&\leq 2C \left(\frac{pd-p}{d-p}\right) \| f^{\frac{d(p-1)}{d-p}} \|_{\frac{p}{p-1}} \| f \|_{D(p)} .
\end{aligned}$$

Observe $\| f^{\frac{pd-p}{d-p}} \|_{\frac{d}{d-1}} = \| f^{\frac{pd}{d-p}} \|_1^{\frac{d-1}{d}}$ and $\| f^{\frac{d(p-1)}{d-p}} \|_{\frac{p}{p-1}} = \| f^{\frac{pd}{d-p}} \|_1^{\frac{p-1}{p}}$. Substituting we obtain $\| f^{\frac{pd}{d-p}} \|_1^{\frac{d-1}{d}} \leq C' \| f^{\frac{pd}{d-p}} \|_1^{\frac{p-1}{p}} \| f \|_{D(p)}$. Dividing this inequality by $\| f^{\frac{pd}{d-p}} \|_1^{\frac{p-1}{p}}$ and noting that $\| f^{\frac{pd}{d-p}} \|_1^{\frac{d-p}{pd}} = (\| f \|_{\frac{pd}{d-p}}^{\frac{pd}{d-p}})^{\frac{d-p}{pd}}$ will yield the claim in the proposition. □

We shall say that a group G satisfies condition $(IS)_d$ if $|A|^{d-1} < C|\partial A|^d$ for all finite subsets A of G and a positive constant C. Varopoulous proves the following proposition on page 224 of [12], also see Chapter 1.4 of [15].

Proposition 5.3. *A finitely generated group G satisfies condition $(IS)_d$ for some $d \geq 1$ if and only if it satisfies condition S_d.*

Our next result will show that each nonzero class in $H^1(G, L^p(G))$, where G is a group with polynomial growth of (precise) degree $d > 2$, can be represented by a function in $L^{p'}(G)$ for some fixed real number $p' > p$.

Theorem 5.4. *Let G be a finitely generated group with polynomial growth of (precise) degree d. If $d > p \geq 2$, then each nonzero class in $H^1(G, L^p(G))$ can be represented by a function from $L^{\frac{pd}{d-p}}(G)$.*

Proof. Varopoulous proves in the papers [13, 14] that G has polynomial growth of (precise) degree d if and only if G satisfies condition $(IS)_d$. Thus G also satisfies condition S_d.

Let 1_G denote the constant function one on G. If $1_G \in (\overline{\mathbb{R}G})_{D^p(G)}$, then there exists a sequence f_n in $\mathbb{R}G$ such that $\| 1_G - f_n \|_{D^p(G)} \to 0$ but $\| f_n \|_{\frac{pd}{d-p}} \not\to 0$ contradicting Proposition 5.2. Hence $(\overline{\mathbb{R}G})_{D^p(G)} \neq D^p(G)$. By Theorem 3.5 each $f \in D^p(G)$ can be represented uniquely by $u + h$, where $u \in (\overline{\mathbb{R}G})_{D^p(G)}$ and $h \in HD^p(G)$. By [8, Corollary 1.10], $HD^p(G) = \mathbb{R}$. Thus nonzero classes in $H^1(G, L^p(G))$ can be represented by functions in $(\overline{\mathbb{R}G})_{D^p(G)} \setminus L^p(G)$. Let $f \in (\overline{\mathbb{R}G})_{D^p(G)}$, so there exists a sequence $\{f_n\}$ in $\mathbb{R}G$ such that $f_n \to f$ in the Banach space $D^p(G)$. Observe that $f_n \to f$ in $D^r(G)$ for $p \leq r < \infty$. By Proposition 5.2 $\{f_n\}$ forms a Cauchy sequence in $L^{\frac{pd}{d-p}}(G)$. Let $\bar{f}$ be the limit of this sequence in $L^{\frac{pd}{d-p}}(G)$. It now follows $f_n \to \bar{f}$ in $D^{\frac{pd}{d-p}}(G)$ since $\| (\bar{f} - f_n) * (s-1) \|_{\frac{pd}{d-p}} \to 0$ as $n \to \infty$ for each $s \in S$. Therefore $\bar{f} = f$ since $f_n \to f$ in $D^{\frac{pd}{d-p}}(G)$. □

Corollary 5.5. *Let $2 < p \in \mathbb{R}$ and let d be an integer greater than p. Each nonzero class in $H^1(\mathbb{Z}^d, L^p(\mathbb{Z}^d))$ can be represented by a function from $L^{\frac{pd}{d-p}}(\mathbb{Z}^d)$.*

6. Some inclusion results

Since $L^p(G) \subseteq L^{p'}(G)$ for $1 < p \leq p' \in \mathbb{R}$ a natural question to ask is how does $H^1(G, L^p(G))$ relate to $H^1(G, L^{p'}(G))$. In this section we will give some answers to this question.

Lemma 6.1. *Let $h \in HD^p(G)$. If $h \in C_0(G)$, then $h = 0$.*

Proof. The set $\{g \mid |h(g)| > \epsilon\}$ is finite for a given $\epsilon > 0$. Thus there exists an $x \in G$ such that $|h(x)| \geq |h(g)|$ for all $g \in G$. It follows $h(x) = h(xs^{-1})$ for all $s \in S$ since $\sum_{s \in S} |h(xs^{-1}) - h(x)|^{p-2} h(xs^{-1}) = \sum_{g \in S} |h(xs^{-1}) - h(x)|^{p-2} h(x)$. We now obtain $h(x) = h(g)$ for all $g \in G$ since the Cayley graph of G is connected. Therefore $h = 0$. □

We now give an inclusion result for nonamenable groups.

Proposition 6.2. *Let G be a finitely generated nonamenable group. If $1 < p \leq p' \in \mathbb{R}$, then $H^1(G, L^p(G)) \subseteq H^1(G, L^{p'}(G))$.*

Proof. Let f represent a nonzero class in $H^1(G, L^p(G))$. Thus $f \in D^p(G) \subseteq D^{p'}(G)$ and $f \notin L^p(G)$. By Theorem 3.5 we can uniquely write $f = u + h$, where $u \in L^p(G)$ and h is a nonconstant element in $HD^p(G)$. By Lemma 6.1 $h \notin C_0(G)$ so it follows $f \notin L^{p'}(G)$. Hence f also represents a nonzero class in $H^1(G, L^{p'}(G))$. □

We now finish this section by giving an example to show the above proposition is not true for amenable groups. Define $f \colon \mathbb{Z} \to \mathbb{R}$ by

$$f(n) = \begin{cases} \frac{1}{\sqrt[p]{n}} & n \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

for some $1 < p \in \mathbb{R}$. Observe that $f \notin L^p(\mathbb{Z})$ but $f \in L^{p'}(\mathbb{Z})$ for $p' > p$. Now

$$\begin{aligned}\sum_{n=1}^{\infty} |(f*(s-1))(n)|^p &= \sum_{n=1}^{\infty} |f(n-1) - f(n)|^p \\ &= 1 + \sum_{n=2}^{\infty} \left| \frac{\sqrt[p]{n} - \sqrt[p]{(n-1)}}{\sqrt[p]{n(n-1)}} \right|^p \\ &\leq 1 + \sum_{n=2}^{\infty} \frac{1}{(n-1)^2}.\end{aligned}$$

Thus $f \in D^p(\mathbb{Z})$ which implies f represents a nonzero class in $H^1(\mathbb{Z}, L^p(\mathbb{Z}))$ but f is in the zero class of $H^1(\mathbb{Z}, L^{p'}(\mathbb{Z}))$ for $p' > p$.

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Department of Mathematics, Eastern Oregon University, One University Boulevard, LaGrande, OR 97850, USA

E-mail address: mpuls@eou.edu