Preliminary Report

Faculty Scholar Program

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The research accomplished during the period supported by the scholars award addressed several of the areas targeted in the proposal. The general topic was recursive algebra, asking recursion theoretic questions in the context of classical algebra. The majority of the effort was directed toward the subjects of degree spectra and identification of questions in recursive commutative algebra.

The *degree spectrum* of an algebraic structure refers to the range of possible Turing degrees which can be associated with a structure or relation under certain initial assumptions.

Questions which received attention include:

1. What is the relationship between the degree of a Boolean algebra B and the degree of a linear order L such that B is effectively isomorphic to $\mathrm{Intalg}(L)$?

2. Given a Boolean algebra B, let $L(B)$ be the set of all linear orders L such that B is isomorphic to $\mathrm{Intalg}(L)$. What can be said about the relationship between the degree of B and the spectra of elements of $L(B)$?

3. For different classes of rings, find algebraic characterizations which imply the existence of computable copies.

4. For a specific class of rings, e.g. Noetherian, what can be said about the degree classes of certain isomorphism types?

5. What can be said about the spectra of ideal membership computability for computable Noetherian rings?

There are several partial responses to the qestions above which can be extracted from the existing literature.

For example, with regard to question 5 above, it is known [B] that every ideal of a computable Noetherian ring has a computable membership relation. However, there is a known example [B] of a computable Noetherian ring with no ideal membership *algorithm.* This means that while the ideals of this ring will all be computable, there is no single uniform method which will address the membership question for all the ideals.

More intuitively, we might say the example illustrates the concept of undecidability, wherein we demonstrate the non-existence of a computer program which might solve a given problem. Undecidability is another common theme in recursion theory as applied to other fields of mathematics.

However, our interest now has to do with studying this "undecidability" in the following way. To say the relation is undecidable is to say that it resides somewhere above the lowest level in a hierarchy of complexity known as the Turing degrees. Now, what can be said about the Turing degrees of the general membership relation? More particularly, what conditions need be imposed on the ring in order to ensure spectra of various types for the membership relations?

This is one example of a line of research which may prove to be fruitful.

The following bibliography is a partial list of literature reviewed in connection with the project.

References

[B] W. Baur: Rekursive Algebren mit Kettenbedingungen, Zeits. Math. Logik Grundl. Math., 20 (1974), 37-46

[D-R] R. Downey, and J. B. Remmel: Questions in Computable Algebra and Combinatorics, Contemporary Mathematics, **257**, 95-125, 2000.

[Ha1] V. Harizanov: Uncountable Degree Spectra, Ann. Pure and Applied Logic, **54**(1991) 255-263.

[Ha2] V. Harizanov: Some effects of Ash-Nerode and other decidability conditions on degree spectra, Ann. Pure and Applied Logic, **55** (1991) 51-65

[Hi] D. R. Hirschfeldt: Degree Spectra of Relations on Computable Structures, Bulletin of Symbolic Logic, vol. 6: 197-212 (2000)

[M] R. Miller: The Δ^0_2 Spectrum of a Linear Order, Journal of Symbolic Logic, vol 66 (2001): 470-486.

[S-H, T] V. Stoltenberg-Hansen, J.V. Tucker: Computable Rings and Fields, in Handbook of Computability Theory, Studies in Logic and the Foundations of Mathematics, vol. 140, Elsevier Science, Amsterdam, 1999.